Eidgenössische Technische Hochschule Zürich

Funktionentheorie

Autumn Semester 2026

Lecture Notes

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References

[1] Imamoglu, Özlem: Vorlesungsnotizen Funktionentheorie. ETH Zürich. 2026

Funktionentheorie

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1 Introdcution

Lec 1

1.1 Motivation

Consider the following example. On $\mathbb{R}$ it is not too difficult to construct a function f that is differentiable n times but not differentiable $n+1$ times. For example,

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}) & x \neq 0, \\ 0 & x = 0 \end{cases}.$$

This function is differentiable but the derivative is not continuous. Integrating, we find a function that is differentiable twice but not three times. In fact, we can construct a function that is differentiable n times but not $n+1$ times for any $n \in \mathbb{N}$ using this method.

In contrast, we'll see that if $f : \mathbb{C} \to \mathbb{C}$ is differentiable once, then it is differentiable infinitely many times.

Another example is that for real numbers, there are functions $f : \mathbb{R} \to \mathbb{R}$ which are infinitely differentiable but not analytic. I.e. the Taylor series at some point x_0 exists, but does not converge to the function in any neighborhood of x_0. For example,

$$x \longmapsto (x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0, \\ 0 & x = 0 \end{cases},$$

where all derivatives are zero so the Taylor series is identically zero.

In contrast, we'll see that a differentiable function $f : \mathbb{C} \to \mathbb{C}$ has a power series representation. So differentiability is equivalent to analyticity.

A further comparison is that there are many examples of functions $f : \mathbb{R} \to \mathbb{R}$ which are C^∞ and bounded, for example

$$f(x) = \sin(x).$$

In contrast, we'll see that if $f : \mathbb{C} \to \mathbb{C}$ is bounded and differentiable, then it is constant.

The robustness of this theory can be used for applications which seem to have nothing to do with complex analysis. Probably the most used examples are evaluating real integrals.

$$\int_0^\infty \cos(t^2)\mathrm{d}t = \int_0^\infty \sin(t^2)\mathrm{d}t = \frac{\sqrt{2\pi}}{4}.$$

Another famous example is the prime number theorem, which states that the number of primes less than x is asymptotic to $\frac{x}{\log(x)}$.

Another classic is to consider a non-zero polynomial $f(x) \in \mathbb{C}[x]$. Then the Fundamental Theorem of Algebra states that f has a zero in $\mathbb{C}$.

Lastly, consider

$$r_4(n) = \#\{(x, y, z, w) \in \mathbb{Z}^4 : x^2 + y^2 + z^2 + w^2 = n\}.$$

Lagrange proved that every positive integer has such a representation, namely $r_4(n) \geq 1$. Jacobi later proved that

$$r_4(n) = 8 \sum_{d | n; 4 \nmid d} d.$$

1.2 Review of Complex Numbers

Recall that the complex numbers are given as

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\},$$

where $i^2 = -1$. We can embed $\mathbb{R}$ into $\mathbb{C}$ via $x \mapsto x + i0$. We also defined

Definition 1.1: Real and Imaginary Parts

For $z = x+iy \in \mathbb{C}$, we define the REAL PART $\Re(z) = x$ and the IMAGINARY PART $\Im(z) = y$.

Furthermore, the COMPLEX CONJUGATE is defined as

$$\overline{z} = x - iy.$$

From this it follows that

Lemma 1.2:

For $z = x + iy \in \mathbb{C}$, we have

$$z \in \mathbb{R} \Leftrightarrow z = \overline{z} \qquad z \in i\mathbb{R} \Leftrightarrow z = -\overline{z}.$$

$\mathbb{C}$ can also be represented by pairs of real numbers in $\mathbb{R}^2$. For example $z = (x, y), w = (u, v) \in \mathbb{R}^2$ where

$$z = w \Leftrightarrow x = u, y = v.$$

Addition can be defined in both notations as

$$z + w = (x + u, y + v) \qquad z + w = (x + w) + i(y + v).$$

Multiplication was a bit more cumbersome. Using the definition of i, we have

$$\begin{aligned} z \cdot w &= (x + iy)(u + iv) \\ &= xu + i^2 yv + i(xv + yu) \\ &= (xu - yv) + i(xv + yu). \end{aligned}$$

So in the pair representation, multiplication is given by

$$(x, y) \cdot (u, v) = (xu - yv, xv + yu).$$

Note that in the pair representation,

$$i \leftrightarrow (0, 1) \quad \text{and} \quad 1 \leftrightarrow (1, 0).$$

The good thing about the pair notation, $(\mathbb{R}^2, +, \cdot)$ satisfies

1. $(\mathbb{R}^2, +)$ is an abelian group with identity $(0, 0)$
2. $(\mathbb{R}^2 \setminus \{(0, 0)\}, \cdot)$ is an abelian group with identity $(1, 0)$
3. The distributive law holds: $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

Together, these properties make $(\mathbb{R}^2, +, \cdot)$ a field. So the complex numbers with the above addition and multiplication form a field.

The additive inverse of $z = (x, y)$ is given by $-z = (-x, -y)$. The multiplicative inverse of $0 \neq z = (x, y)$ is given by

$$z^{-1} = \frac{\overline{z}}{|z|^2} = \frac{(x, -y)}{x^2 + y^2} = \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2}\right).$$

Here, $|z| \in \mathbb{R}$ is the MODULUS or NORM of z, defined as

$$|z| = \sqrt{x^2 + y^2}.$$

We can also consider the POLAR COORDINATES of z. Any point on the plane $\mathbb{R}^2$ can be represented by a distance r from the origin and an angle θ from the positive x-axis. Here,

$$z = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}.$$

Beware that the representation is not unique, since θ can be replaced by $\theta + 2\pi k$ for any $k \in \mathbb{Z}$, unless we restrict θ to a certain interval. The standard is to take $\theta \in (-\pi, \pi]$. θ is called the ARGUMENT of z, denoted $\arg(z)$, and r is the modulus $|z|$. For the argument in $(-\pi, \pi]$, we write $\operatorname{Arg}(z)$ and call it PRINCIPAL ARGUMENT.

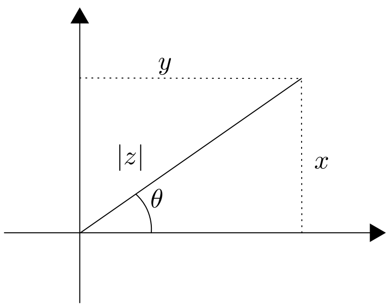

Figure 1: Polar coordinates of a complex number z

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The principal argument of $z = x + iy \neq 0$ we have

$$\operatorname{Arg}(z) = \begin{cases} \arcsin(\frac{y}{|z|}) & x \geq 0 \\ \pi - \arcsin(\frac{y}{|z|}) & x < 0, y \geq 0 \\ -\pi - \arcsin(\frac{y}{|z|}) & x < 0, y < 0 \end{cases}.$$

Here, for $t \in [-1, 1]$, $\arcsin(t)$ is the unique number u in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin(u) = t$.

Notice, that

$$\arg(z^-1) = -\arg(z) \quad \text{and} \quad \arg(zw) = \arg(z) + \arg(w).$$

But, it is not always the case if we use the principal argument instead. For example

$$\operatorname{Arg}(-\frac{1}{2}) = \pi \neq -\operatorname{Arg}(-2) = -\pi.$$

Lemma 1.3:

Let $z, w \in \mathbb{C}$. Then

- $|z| = 0 \Leftrightarrow z = 0$
- $||z| - |w|| \leq |z + w| \leq |z| + |w|$
- $|zw| = |z||w|$
- $|\overline{z}| = |z|$
- $|\Re(z)| \leq |z|$ and $|\Im(z)| \leq |z|$

If we have two numbers $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ in polar form, then

$$z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}.$$

$\mathbb{C}$ also has a representation in terms of 2×2 matrices. For $z = a + ib$, let

$$Z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

If $w = c + id$, then

$$ZW = \begin{pmatrix} ac - bd & -(bc + ad) \\ bc + ad & ac - bd \end{pmatrix}.$$

So we can do the multiplication of complex numbers using matrix multiplication.

We can also write the matrix representation as

$$Z = \Re(z)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \Im(z)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In polar form, $z = re^{i\theta}$, we have

$$Z = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Notice that the second matrix lives in $SO(2)$, the group of rotations in $\mathbb{R}^2$.

1.3 Notations from Analysis

We like to recall some definitions from analysis regarding open and closed sets.

Definition 1.4: Discs

The OPEN DISC of radius r centered at z is the set

$$D_r(z) = \{w \in \mathbb{C} : |w - z| < r\}.$$

The CLOSED DISC of radius r centered at z is the set

$$\overline{D}_r(z) = \{w \in \mathbb{C} : |w - z| \leq r\}.$$

The boundary circle is the set

$$C_r(z) = \{w \in \mathbb{C} : |w - z| = r\}.$$

Definition 1.5: Connected and Disconnected Sets

A subset $A \subset \mathbb{C}$ is called DISCONNECTED if there exists two open sets $U, V \subset \mathbb{C}$ such that

$$U \cap V = \varnothing, \quad A \cap U \neq \varnothing, \quad A \cap V \neq \varnothing, \quad A \subset U \cup V.$$

A set is called CONNECTED if it is not disconnected. A connected open subset $U \neq \varnothing$ of $\mathbb{C}$ is called a REGION or a DOMAIN.

1.4 Holomorphic Functions

We now want to define what it means for a complex function to be differentiable.

Definition 1.6: Holomorphic functions

Let $\Omega \subset \mathbb{C}$ be an open and $f : \Omega \to \mathbb{C}$ a function.

f is called DIFFERENTIABLE or HOLOMORPHIC at $z_0 \in \Omega$ if

$$\lim_{z \to z_0, z \neq z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0, h \neq 0} \frac{f(z+h) - f(z)}{h}.$$

If f is differentiable $\forall z \in \Omega$, it is holomorphic on Ω. If $\Omega = \mathbb{C}$ and f is holomorphic on $\mathbb{C}$ then f is called ENTIRE.

The value of the limit is denoted by $f'(z_0)$.

Example 1.7:

Consider $f : \mathbb{C} \to \mathbb{C}, z \mapsto z$. Then f is entire. Indeed,

$$\lim_{h \to 0, h \neq 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0, h \neq 0} \frac{z + h - z}{h} = 1.$$

Proposition 1.8: Properties of holomorphic functions

1. Let $\mathcal{H}(\Omega)$ be the set of holomorphic functions on Ω. Then $\mathcal{H}(\Omega)$ is a vector space over $\mathbb{C}$.

Here, $O : z \mapsto 0$ is the 0-element.

2. If $f, g \in \mathcal{H}(\Omega)$ then so is $f \cdot g$. Also, we have

$$(fg)'(z_0) = f'(z_0)g(z_0) + g'(z_0)f(z_0).$$

3. Lastly if $g(z_0) \neq 0$, for any $z_0 \in \Omega$, then f/g is holomorphic at z_0 and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - g'(z_0)f(z_0)}{g(z_0)^2}.$$

4. If $f : \Omega \to U$ and $g : U \to \mathbb{C}$ are holomorphic, then $g \circ f$ is holomorphic and

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

Proof. The proof is the same as in real analysis. We will thus only show the proof of the last statement.

Let $z_0 \in \Omega$ and let $f(z_0) = w_0 \in U$. Consider

$$F : \Omega \to \mathbb{C}, G : U \to \mathbb{C}.$$

where

$$F(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} & z \neq z_0 \\ f'(z_0) & z = z_0 \end{cases}, \quad G(z) = \begin{cases} \frac{g(w)-g(w_0)}{w-w_0} & w \neq w_0 \\ g'(w_0) & w = w_0 \end{cases}.$$

Since f and g are holomorphic, F and G are continuous. Hence, $G \circ f$ is continuous at z_0.[1] Now,

$$\frac{(g \circ f)(z) - (g \circ f)(z_0)}{z - z_0} = \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)}.$$

Using $f(z_0) = w_0$ we can write this as

$$\frac{g(f(z)) - g(w_0)}{f(z) - w_0} \cdot \frac{f(z) - f(z_0)}{z - z_0} = G(f(z))F(z).$$

Note that if $f(z) = w_0$ we get that $F(z) = \frac{w_0 - f(z_0)}{z - z_0} = 0$. Hence, taking the limit $z \to z_0$ and using continuity, we get that

$$\begin{aligned} \lim_{z \to z_0, z \neq z_0} \frac{(g \circ f)(z) - (g \circ f)(z_0)}{z - z_0} &= \lim_{z \to z_0, z \neq z_0} G(f(z))F(z) \\ &= G(f(z_0))F(z_0) \\ &= g'(f(z_0))f'(z_0). \end{aligned}$$

□

Remark

If $f : \Omega \to \mathbb{C}$ is differentiable at $z_0 \in \Omega$, then $\exists$ a complex number $c \in \mathbb{C}$ such that

$$f(z) = f(z_0) + c(z - z_0) + o(|z - z_0|) \quad \text{as } z \to z_0.$$

Here, for c we write $f'(z_0)$.

Example 1.9: Polynomials are entire

Using $f(z) = z$ entire and Proposition 1.8, we get that any polynomial is entire. If $g(z) = z^n$, then

$$g'(z) = nz^{n-1}.$$

Example 1.10: Real and complex differentiability

Let $f(z) = \overline{z}$. Then

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\overline{z_0 + h} - \overline{z_0}}{h} = \frac{\overline{h}}{h}.$$

If we choose $h = t, t \in \mathbb{R}$, then $\frac{\overline{h}}{h} = 1$. If we choose $h = it, t \in \mathbb{R}$, then $\frac{\overline{h}}{h} = -1$. Hence, the limit does not exist and f is not holomorphic.

If we look at f as a function from $\mathbb{R}^2 \to \mathbb{R}^2$, then $\tilde{f}(x, y) = (x, -y)$. Then $\tilde{f}$ is differentiable in the real sense and the Jacobian is given by

$$D\tilde{f}(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall, that a function $f : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (u, v)$ is differentiable at (x_0, y_0) if there exists a linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\lim_{(x,y) \to (x_0, y_0)} \frac{|f(x, y) - f(x_0, y_0) - L(x - x_0, y - y_0)|}{|(x - x_0, y - y_0)|} = 0.$$

The linear map L is unique and called the DIFFERENTIAL Df. In the standard basis, the linear map L has a matrix representation which is called the JACOBIAN of f. It is given by

$$Jf(x_0, y_0) = \begin{pmatrix} \frac{\partial u}{\partial x}(x_0, y_0) & \frac{\partial u}{\partial y}(x_0, y_0) \\ \frac{\partial v}{\partial x}(x_0, y_0) & \frac{\partial v}{\partial y}(x_0, y_0) \end{pmatrix}.$$

[1] f is differentiable at z_0 so continuous at z_0.

Recall from linear algebra, that we can view $\mathbb{C}$ as a vector space over $\mathbb{C}$ with dimension 1 with basis $\{1\}$. Alternatively we can view $\mathbb{C}$ as a vector space over $\mathbb{R}$ with dimension 2 with basis $\{1, i\}$.

A map $T : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-linear if $Tz = \lambda z$ for some $\lambda \in \mathbb{C}$. It is $\mathbb{R}$-linear if

$$T(x + iy) = xT(1) + yT(i) \quad \forall x, y \in \mathbb{R}.$$

Using $x = \frac{z+\overline{z}}{2}$ and $y = \frac{z-\overline{z}}{2i}$, we can write

$$T(z) = \underbrace{\frac{T(1) - iT(i)}{2}}_{:=\lambda} z + \underbrace{\frac{T(1) + iT(i)}{2}}_{:=\mu} \overline{z}.$$

Since T is linear, the coefficient of $\overline{z}$ must be 0. So every $\mathbb{C}$-linear map is $\mathbb{R}$-linear if we set $\mu = 0$. However an $\mathbb{R}$-linear map is only $\mathbb{C}$-linear if $\mu = 0$, so

$$T(i) = iT(1).$$

If $T(1) = a + ib$ and $T(i) = c + id$, then the condition $T(i) = iT(1)$ is equivalent to

$$a = d \quad \text{and} \quad b = -c.$$

Every $\mathbb{R}$-linear map $\mathbb{R}^2 \to \mathbb{R}^2$ is given by a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If this $\mathbb{R}$-linear map is also $\mathbb{C}$-linear, then on the matrix we have $a = d$ and $b = -c$. Hence, the matrix is of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

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